B.Sc.- part- II (Sub) Diff. Calculus. 01-07-2020

Problems based on Leibnitz's Theorem.

Q.1 If $y = x^2 e^{ax}$, find y_n.

Soln: We have, $y = x^2 e^{ax} = (e^{ax})(x^2)$

Let $u = e^{ax}$

$\therefore u_n = a^n e^{ax}$, $u_{n-1} = a^{n-1} e^{ax}$, $u_{n-2} = a^{n-2} e^{ax}$, etc.

Again, Let $v = x^2$ $\therefore v_1 = 2x$, $v_2 = 2$, $v_3 = 0$, $v_4 = 0$ etc.

$\therefore y = uv$

Differentiating both sides, ~~w.r.t. x, we get~~ n times with respect to x, by Leibnitz's theorem, We get

$$y_n = (uv)_n$$

$$\Rightarrow y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_n u v_n$$

$$\Rightarrow y_n = a^n e^{ax} x^2 + n a^{n-1} e^{ax}(2x) + \frac{n(n-1)}{1.2} a^{n-2} e^{ax}(2) + \text{other terms which vanish.}$$

Hence, $y_n = e^{ax} a^{n-2} [x^2 a^2 + 2nax + n(n-1)]$

Solved

Q.2. Show that $\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n \lfloor n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right)$

Soln: Let $U = \frac{1}{x}$ and $V = \log x$

Then, $U_n = \frac{(-1)^n \lfloor n}{x^{n+1}}$, $U_{n-1} = \frac{(-1)^{n-1} \lfloor n-1}{x^n} = -\frac{(-1)^n \lfloor n-1}{x^n}$

$U_{n-2} = \frac{(-1)^{n-1} \lfloor n-2}{x^{n-1}} = \frac{(-1)^n \lfloor n-2}{x^{n-1}}$

$U_{n-3} = \frac{(-1)^{n-3} \lfloor n-3}{x^{n-2}} = -\frac{(-1)^n \lfloor n-3}{x^{n-2}}$, etc.

and $V_1 = \frac{1}{x}$, $V_2 = -\frac{1}{x^2}$, $V_3 = \frac{2}{x^3}$, ...

$\therefore V_n = -\frac{(-1)^n \lfloor n-1}{x^n}$

Now, L.H.S $= \frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{d^n}{dx^n}(UV) = (UV)_n$

$= U_nV + nU_{n-1}V_1 + \frac{n(n-1)}{1\cdot 2}U_{n-2}V_2 + \cdots + UV_n$

by Leibnitz's Theorem

$= \frac{(-1)^n \lfloor n}{x^{n+1}}\log x - n\frac{(-1)^n \lfloor n-1}{x^n}\cdot\frac{1}{x} - \frac{n(n-1)}{1\cdot 2}\cdot\frac{(-1)^n \lfloor n-2}{x^{n-1}}\cdot\frac{1}{x^2}$

$- \cdots - \frac{1}{x}\cdot\frac{(-1)^n \lfloor n-1}{x^n}$

$= \frac{(-1)^n \lfloor n}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right) =$ R.H.S

Proved